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THE EHRENFEST CHAIN AS AN APPROXIMATION TO THE O-U PROCESS.(U)
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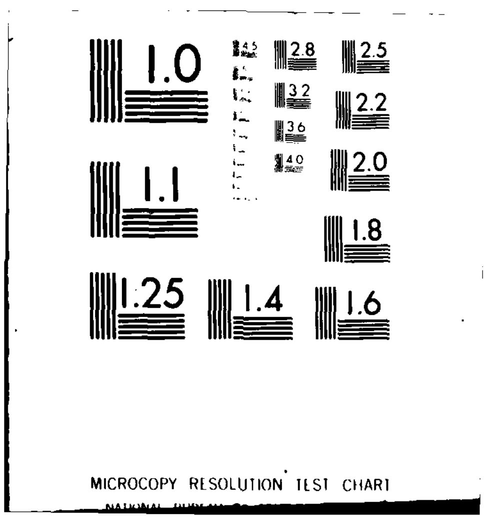
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THE EHRENFEST CHAIN AS AN APPROXIMATION TO THE O-U PROCESS

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A sequence of birth-death processes successively approximating the Ornstein-Uhlenbeck process is described. These Ehrenfest approximations have advantages of tractability and computerability required for subsequent development of certain diffusion processes related to the Ornstein-Uhlenbeck process of meteorological interest.

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§0. Introduction and Summary

Because of analytical and numerical difficulty, effort is often made to approximate a birth-death process by a diffusion process. Such approximation is usually weakened by the absence of either numerical comparison or analytical error bound. The present small paper was motivated by a need to replace the Ornstein-Uhlenbeck (O-U) process by a certain finite birth-death process, the Ehrenfest process as described below. For these two processes, a numerical comparison is possible. On the one hand, extensive numerical results for the O-U process are available [8,9]. Simultaneously, algorithms and software available for birth-death processes [5,7] extend, with some further analytical and numerical effort, to the approximating Ehrenfest process.

The Ehrenfest process, a finite Markov chain of some practical interest, arises from the sum of K independent identical chains $J_j(t)$, each on state space $\{0,1\}$ and governed by transition rates $\nu_{01} = \nu_{10} = 1/2$. The Markov chain $N_K(t) = \sum_{j=1}^{K} J_j(t)$ has transition rates $\nu_{n,n+1} = \frac{1}{2}(K-n)$, $0 \le n \le K-1$, and $\nu_{n,n-1} = \frac{1}{2}n$, $1 \le n \le K$. Consequently, the local growth rate of the variance $\nu_{n,n+1} - \nu_{n,n-1} = \frac{K}{2} - n$. For the associated stationary chain $N_{KS}(t)$, one $\mathrm{cov}[N_{KS}(t), N_{KS}(t+\tau)] = \frac{K}{4}e^{-\tau}$ and asymptotic normality.

These simple properties suggest that the sequence of processes $X_V(t) = \sqrt{\frac{2}{V}}\, N_{2V}(t) - \sqrt{2V}$ converges in law to the Ornstein-Uhlenbeck process as $V \to \infty$, since such O-U processes are characterized by their Markov property, normal distribution and exponential covariance function. The object of this research note is to establish this convergence in law,

to develop systematically the properties of $N_{2V}(t)$, and to quantify its behavior.

Of particular interest is the evaluation of the quality of the chain $X_V(t)$ as an approximation to the O-U process. Even though the O-U process $X_\infty(t)$ is quite tractable [8,9], simple related processes such as $Y_\theta(t) = \theta \int_{-\infty}^{t} e^{-\theta(t-t')} X_\infty(t')dt'$ are intractable. The latter process $Y_\theta(t)$ describes an averaged O-U process with averaging time θ^{-1}. Evaluation analytically of the passage time distributions for $Y_\theta(t)$ and related distribution of the maximum of $Y_\theta(t)$ over an interval has defeated effort to date.

Recently the authors have developed an algorithmic framework [10] for quantifying the behavior of finite bivariate Markov chains $[N_1(t), N_2(t)]$ where the marginal process $N_2(t)$ is lattice-continuous. The bivariate Markov process $[X_\infty(t), Y_\theta(t)]$ may be approximated by a finite bivariate Markov chain $[X_V(t), Y_{\theta V}(t)]$ whose sequence converges in law to $[X_\infty(t), Y_\theta(t)]$. It is expected that the use of $X_V(t)$ in conjunction with that framework will permit evaluation of the distributions needed for $Y(t)$.

The structure of this paper is as follows. Following Karlin and McGregor [2], notation is established for the spectral representation of birth-death processes in §1. The specific form appropriate to the Ehrenfest chain $N_{2V}(t)$ is developed in §2. Related first passage time density structure is described in §3. In §4, the first passage time structure for the Ehrenfest chain $N_{2V}(t)$ is studied in detail. In particular, for the first passage time T_{0V} of $N_{2V}(t)$ from 0 to V, we show

that $2T_{0V}$ - log V converges in law to a distribution conjugate to the extreme-value distribution. The convergence in law of $X_V(t)$ to the O-U process and related results are established in §5. Finally, in §6, the numerical results are exhibited. In particular, the ergodic probabilities and the first passage time distributions of $X_V(t)$ for V = 10, 25 and 40 and those of the O-U process are compared.

§1. The spectral representation of birth-death processes

A birth-death process $N(t)$ is a Markov chain in continuous time on $n = \{0,1,2,\ldots\}$ governed by a transition rate matrix $\underline{\underline{\nu}}$ where

$$\nu_{mn} = \begin{cases} \lambda_m > 0 & n = m+1 \ , \quad m = 0,1,2,\ldots \\ \mu_m > 0 & n = m-1 \ , \quad m = 1,2,\ldots \\ 0 & |n-m| > 1 \end{cases} .$$

Let $\underline{\underline{Q}} = \underline{\underline{\nu}} - \underline{\underline{\nu}}_D$ where $\underline{\underline{\nu}}_D$ is a diagonal matrix with m-th diagonal element equal to $\sum_{n \epsilon N} \nu_{mn}$. The transition probability matrix $\underline{\underline{P}}(t)$ is then given by [5]

$$(1.1) \qquad \underline{\underline{P}}(t) = e^{\underline{\underline{Q}}t}$$

where $\underline{\underline{Q}}^0 \overset{\text{def}}{=} \underline{\underline{I}}$. The matrix $\underline{\underline{Q}}$ is called the infinitesimal generator of the chain $N(t)$.

In a series of papers [2], [3], [4], Karlin and McGregor analyze the spectral representation of $\underline{\underline{P}}(t)$ and use the results to study various probabilistic quantities. In this section we briefly describe their results and establish notation.

The infinitesimal generator $\underline{\underline{Q}}$ has a vector eigenfunction $\underline{y}(x) = [y_n(x)]_{n\epsilon N}$ with associated eigenvalue $-x$, i.e.,

$$(1.2) \qquad \underline{\underline{Q}}\underline{y}(x) = -x\underline{y}(x) ,$$

or componentwise one has with $y_0(x) = 1$,

$$(1.3) \qquad \begin{cases} -\lambda_0 y_0(x) + \lambda_0 y_1(x) = -x\, y_0(x) \\ \mu_n y_{n-1}(x) - (\lambda_n + \mu_n) y_n(x) + \lambda_n y_{n+1}(x) = -x\, y_n(x) , \quad n \geq 1 \end{cases}$$

We note from (1.3) that $y_n(x)$ is a polynomial of degree n with leading coefficient $(\prod_{j=0}^{n-1} \lambda_j)^{-1}$. Let $\underline{f}(x,t)$ be a vector function defined by

$$(1.4) \qquad \underline{f}(x,t) = \underline{\underline{P}}(t)\underline{y}(x) \quad .$$

From the Kolmogorov forward equations, one then has

$$\frac{\partial}{\partial t} \underline{f}(x,t) = \frac{d}{dt} \underline{\underline{P}}(t) \cdot \underline{y}(x) = \underline{\underline{P}}(t)\underline{\underline{Q}}\underline{y}(x) = -x\underline{\underline{P}}(t)\underline{y}(x) \quad ,$$

i.e.,

$$(1.5) \qquad \frac{\partial}{\partial t} \underline{f}(x,t) = -x\underline{f}(x,t) \quad .$$

Since $\underline{f}(x, 0+) = \underline{y}(x)$, the partial differential equation (1.5) is solved by

$$(1.6) \qquad \underline{f}(x,t) = e^{-xt}\underline{y}(x) \quad .$$

Equations (1.4) and (1.6) imply that $\underline{f}(x,t) = \underline{\underline{P}}(t)\underline{y}(x) = e^{-xt}\underline{y}(x)$, i.e.,

$$(1.7) \qquad \sum_{n \in N} p_{mn}(t) y_n(x) = e^{-xt} y_m(x) \;, \quad m \in N \quad .$$

It is known [3] that there exists a measure $\psi(x)$ on $[0,\infty)$ such that $\{y_n(x)\}_{n\in N}$ becomes a set of orthogonal polynomials with respect to $\psi(x)$. One has

$$(1.8) \qquad \int_0^\infty y_m(x)y_n(x)d\psi(x) = \delta_{mn}/\pi_n \ , \quad m,n \in N$$

where π_n are the potential coefficients [15] with $\pi_0 = 1$ and $\pi_n = \prod_{j=0}^{n-1} \lambda_j / \prod_{j=1}^{n} \mu_j$. From (1.7), $\{p_{mn}(t)\}_{n\in N}$ may be recognized as the generalized Fourier coefficients of $f_m(x,t)$ associated with the orthogonal polynomials $\{y_n(x)\}_{n\in N}$ for each $m \in N$. One then has, from (1.7),

$$(1.9) \qquad p_{mn}(t) = \pi_n \int_0^\infty e^{-xt} y_m(x)y_n(x)d\psi(x) \ , \quad m,n \in N \ .$$

§2. The spectral representation of the Ehrenfest process

Consider 2V independent and identical Markov chains $J_j(t)$ in continuous time on $\{0,1\}$, governed by the transition rate matrix $\underline{\underline{\nu}} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$. Let $\underline{\underline{q}}(t)$ be the transition probability matrix of $J_j(t)$. One finds that

$$(2.1) \qquad \underline{\underline{q}}(t) = \begin{bmatrix} q_{00}(t) & q_{01}(t) \\ q_{10}(t) & q_{11}(t) \end{bmatrix} = \begin{bmatrix} f(t) & g(t) \\ g(t) & f(t) \end{bmatrix}$$

where $f(t) = \frac{1}{2}(1 + e^{-t})$ and $g(t) = \frac{1}{2}(1 - e^{-t})$. For future reference we define the generating functions

$$(2.2) \qquad \alpha_0(t,u) = q_{00}(t) + q_{01}(t)u = f(t) + g(t)u$$

and

$$(2.3) \qquad \alpha_1(t,u) = q_{10}(t) + q_{11}(t)u = q(t) + f(t)u \quad .$$

Let $N_{2V}(t) = \sum_{j=1}^{2V} J_j(t)$. Then $N_{2V}(t)$ is a birth-death process on $N = \{0,1,\ldots,2V\}$, governed by the transition rates

$$(2.4) \qquad \lambda_n = \frac{1}{2}(2V-n) \ ; \quad \mu_n = \frac{n}{2} , \quad n \in N \quad .$$

We note that

$$(2.5) \qquad \nu_n = \lambda_n + \mu_n = V , \quad n \in N \quad .$$

This birth-death process may be called the Ehrenfest process [1]. Let

$\underline{\underline{P}}(t)$ be the transition probability matrix of $N_{2V}(t)$ and define the generating function

$$\beta_m(t,u) = \sum_{k=0}^{2V} p_{mk}(t)u^k \ , \quad m \in N \quad . \tag{2.6}$$

From the independence of the $J_j(t)$, one has

$$\beta_m(t,u) = \alpha_0(t,u)^{2V-m}\alpha_1(t,u)^m = \{f(t) + g(t)u\}^{2V-m}\{g(t) + f(t)u\}^m \ . \tag{2.7}$$

It can be readily seen from (2.7) that $p_{mn}(t)$ is a linear combination of $\{e^{-kt}\}$, $0 \le k \le 2V$. Karlin and McGregor [ ] identify the orthogonal polynomials and associated discrete measure described in §1 for the Ehrenfest process and give the following spectral representation.

$$y_n(x) = \frac{1}{\binom{2V}{n}} \sum_{j=0}^{2V} \binom{2V-x}{n-j}\binom{x}{j}(-1)^j \ , \quad n \in N \ , \tag{2.8}$$

$$d\psi(x) = \binom{2V}{x} 2^{-2V} \ , \quad x = 0,1,\ldots,2V \ , \tag{2.9}$$

and

$$P_{mn}(t) = \binom{2V}{n} 2^{-2V} \sum_{j=0}^{2V} \binom{2V}{j} y_m(j) y_n(j) e^{-jt} \ . \tag{2.10}$$

The polynomials $(y_n(x))$ are called the Krawtchouk polynomials.

The ergodic distribution of $J_j(t)$ is $(\frac{1}{2}, \frac{1}{2})$. Therefore the ergodic distribution $\underline{e}^T = [e_0,\ldots,e_{2V}]$ of the Ehrenfest process is given by

$$(2.11) \qquad e_n = \binom{2V}{n} 2^{-2V}, \quad n \in N$$

from the independence of $J_j(t)$. We notice that

$$(2.12) \qquad p_{mn}(t) = \binom{2V}{n} 2^{-2V} \sum_{j=0}^{2V} \binom{2V}{j} y_m(j) y_n(j) e^{-jt} \to e_n$$

as $t \to \infty$, as expected.

§3. The first passage time and the historical maximum of birth-death processes

Let $N_K(t)$ be a birth-death process on $n = \{0,1,\ldots,K\}$ $(K \le \infty)$. Define the first passage time T_{mn} of $N_K(t)$ from state m to state n by

$$(3.1) \qquad T_{mn} = \inf\{t' : N_K(t') = n \mid N(0) = m\} \quad .$$

Let $s_{mn}(t)$ be the p.d.f. of T_{mn} with Laplace transform $\sigma_{mn}(s)$. For notational convenience we denote $T_{m,m+1}$ by T_m^+ and correspondingly $s_m^+(t)$ and $\sigma_m^+(s)$. From a probabilistic argument [5], one has the following consistency relations

$$(3.2) \qquad \sigma_n^+(s) = \lambda_n[s + \nu_n - \mu_n\sigma_{n-1}^+(s)]^{-1} \quad , \quad n = 1,2,\ldots,K$$

with $\sigma_0^+(s) = \dfrac{\lambda_0}{s + \lambda_0}$. Let $\{g_n(s)\}$ be the polynomials of order n defined by

$$(3.3) \qquad \sigma_{0n}(s) = \frac{1}{g_n(s)} \quad , \quad n = 1,2,\ldots,K \; ; \quad g_0(s) = 1 \quad .$$

Since $\sigma_{0n}(s) = \sigma_{0n-1}(s)\sigma_{n-1}^+(s)$, we have from (3.2)

$$(3.4) \qquad g_{n+1}(s) = \frac{1}{\lambda_m}\,[(s + \nu_n)g_n(s) - \mu_n g_{n-1}(s)] \quad , \quad n = 0,1,2,\ldots,K \quad ,$$

where $g_{-1}(s) = 0$. As shown in [ ], the polynomials $g_n(x)$ are related to the orthogonal polynomials $y_n(x)$ given in §1 by

$$(3.5) \qquad y_n(x) = g_n(-x) \quad .$$

We notice, from (3.3), that

$$\sigma_n^+(s) = g_n(s)/g_{n+1}(s) \quad . \tag{3.6}$$

From (3.5), $\{g_n(s)\}$ are orthogonal polynomials and hence the zeros of $g_n(s)$ are distinct, and the zeros of any two successive polynomials interleave. *Furthermore, the zeros are negative. Therefore* $\sigma_n^+(s)$ *is* of the form

$$\sigma_n^+(s) = \sum_{j=0}^{n} p_{nj} \frac{\alpha_{nj}}{s + \alpha_{nj}} \tag{3.7}$$

where $p_{nj} > 0$, $\sum_{j=0}^{n} p_{nj} = 1$ and $-\alpha_{nj}$ are the zeros of $g_{n+1}(s)$. Hence $s_n^+(t)$ is completely monotone. A more detailed discussion can be found in Keilson [ ]. The downward passage time $T_{n,n-1} = T_n^-$ can be treated similarly when $K < \infty$.

Let $M_K(n_0, \theta)$ be the historical maximum of $N_K(t)$ in the interval $[0,\theta]$ given that $N_K(0) = n_0$, i.e.,

$$M_K(n_0, \theta) = \max_{0 \le t \le \theta} \{N_K(t) | N_K(0) = n_0\} \quad . \tag{3.8}$$

We observe the following dual relation between the first passage time T_{n_0n} $(n_0 < n)$ and $M_K(n_0, \theta)$.

$$F_{n_0,\theta}(n) = P[M_K(n_0, \theta) \le n] = P[T_{n_0n} > \theta] = \bar{S}_{n_0n}(\theta) \quad .$$

Hence one has for integers n

$$(3.9) \qquad F_{n_0\theta}(n) = \begin{cases} 0 & n_0 \geq n \\ \bar{S}_{n_0 n}(\theta) & n_0 < n \end{cases} .$$

For the corresponding stationary process $N_{KS}(t)$, the c.d.f. of the historical maximum is given by

$$(3.10) \qquad F_\theta(n) = \sum_{m<n} e_n \bar{S}_{mn}(\theta)$$

where $\underline{e}^T = [e_n]$ is the stationary distribution of $N_{KS}(t)$.

§4. The first passage time structure of the Ehrenfest process

As we saw in §2, the Ehrenfest process $N_{2V}(t)$ is a birth-death process on $\{0,1,\ldots,2V\}$ governed by transition rates $\lambda_n = (2V-n/2$, $0 \le n \le 2V-1$, and $\mu_n = n/2$, $1 \le n \le 2V$. The recursion (3.4) then becomes

$$(4.1) \qquad g_{n+1}(s) = \frac{2}{2V-n}\,[(s+V)g_n(s) - \frac{n}{2}\,g_{n-1}(s)], \quad 0 \le n \le 2V \quad ,$$

with $g_{-1}(s) = 0$ and $g_0(s) = 1$. From (2.8) and (3.5), $g_n(s)$ are given explicitly by

$$(4.2) \qquad g_n(s) = -\frac{1}{\binom{2V}{n}} \sum_{n=0}^{2V} \binom{2V+s}{n-j}\binom{-s}{j}(-1)^j \;, \quad 0 \le n \le 2V \quad .$$

In order to evaluate the first passage times $s_{mn}(t)$ $(m<n)$ with Laplace transform $\sigma_{mn}(s) = \sigma_m^+(s)\ldots\ldots\sigma_{n-1}^+(s) = g_m(s)/g_n(s)$, the zeros of $g_n(s)$ are needed. The amount of effort required for the zero search is considerably reduced by the following observations.

Theorem 4.1

Let $h_n(s) = g_n(s-V)$. Then $h_n(s) = (-1)^n h_n(-s)$, $n \ge 0$, i.e.,

$$h_n(s) \text{ is } \begin{cases} \text{odd when } n \text{ is odd,} \\ \text{even when } n \text{ is even.} \end{cases}$$

Proof

Eq. (4.1) can be written in terms of $h_n(s)$ by

$$h_{n+1}(s) = \frac{2}{2V-n} [sh_n(s) - \frac{n}{2} h_{n-1}(s)]$$

and the result follows by induction on n. $\square$

Corollary 4.2

(a) $g_n(-x) = 0 \Rightarrow g_n(x - 2V) = 0$

(b) n: odd $\Rightarrow g_n(-V) = 0$.

Proof

This follows immediately from Theorem 4.1. $\square$

Theorem 4.1 implies that the zeros of $h_n(s)$ are symmetric about zero and correspondingly, as in Corollary 4.2, the zeros of $g_n(x)$ are symmetric about $-V$. Hence we need to find only $[(n-1)/2]$ zeros, where $[x]$ is the minimum integer which is greater than or equal to x. Furthermore, there are only $1 + [(n-1)/2]$ terms in each $h_n(s)$ while $g_n(s)$ has $(n+1)$ terms. We therefore reduce computation time approximately by a factor of 4. We note that similar results are available for general birth-death processes whenever $\nu_n = \lambda_n + \mu_n = \nu$ is constant.

We next show that $g_V(s)$ has negative odd integers as its roots. A preliminary lemma is needed.

Lemma 4.3

$g_n(-m) = g_m(-n)$, $0 \le m,n \le 2V$, m,n integers .

Proof

This is a known result [ ]. A simple proof is as follows.

$$\frac{\binom{2V-n}{m-j}\binom{n}{j}}{\binom{2V}{m}} = \frac{(2V-n)!}{(m-j)!(2V-n-m+j)!} \cdot \frac{n!}{j!(n-j)!} \cdot \frac{m!(2V-m)!}{(2V)!}$$

$$= \frac{\binom{2V-m}{n-j}\binom{m}{j}}{\binom{2V}{n}}$$

and the result follows from (4.2). □

Theorem 4.4

$$g_V(s) = \prod_{j=1}^{V} (s+2j-1)/(2j-1) \quad .$$

Proof

Corollary 4.2 (b) states that $g_n(-V) = 0$ for n odd. The theorem is then immediate from Lemma 4.3. □

Eq. (3.3) along with Theorem 4.4 enables one to write the Laplace transform of the first passage time T_{0V} from 0 to V.

$$(4.3) \qquad \sigma_{0V}(s) = \prod_{j=1}^{V} \frac{2j-1}{s+2j-1} = \sum_{j=1}^{V} c_j \frac{2j-1}{s+2j-1}$$

where $c_j = \prod_{\substack{k=1 \\ k \neq j}}^{V} \frac{2k-1}{2(k-j)} = \frac{V}{2^{2V-1}} \binom{2V}{V}\binom{V-1}{j-1}(-1)^{j-1} \cdot \frac{1}{2j-1}$.

The limiting distribution of T_{0V} can now be evaluated. We refer the reader to Keilson [6] for the definition of the conjugate distribution.

Theorem 4.5

$2T_{0V} - \log V \xrightarrow{d} G$ as $V \to +\infty$ where G corresponds to the distribution conjugate to the extreme-value distribution with p.d.f.

$$g(\tau) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}\tau} e^{-e^{-\tau}} \quad , \quad -\infty < \tau < \infty \quad .$$

Proof

The p.d.f. of $2T_{0V} - \log V$ has the Laplace transform

$$V^s \sigma_{0V}(2s) = V^s \prod_{j=1}^{V} \frac{2j-1}{2s+2j-1} \quad .$$

By simple algebra this can be rewritten as

$$(4.4) \qquad V^s \sigma_{0V}(2s) = \frac{V^{\frac{1}{2}}}{2^{2V}} \binom{2V}{V} \{V^{s-\frac{1}{2}} \prod_{j=1}^{V} \frac{j}{s-\frac{1}{2}+j}\} \quad .$$

The factor inside braces converges to $\Gamma(s+\frac{1}{2})$ as $V \to +\infty$, while the rest converges to $1/\sqrt{\pi} = 1/\Gamma(\frac{1}{2})$, i.e.,

$$V^s \sigma_{0V}(2s) \to \frac{\Gamma(s+\frac{1}{2})}{\Gamma(\frac{1}{2})} \quad \text{as} \quad V \to \infty \quad .$$

It is known that $\Gamma(1+s)$ is the Laplace transform of the extreme distribution with p.d.f. $e^{-\tau} e^{-e^{-\tau}}$, $-\infty < \tau < \infty$, and thus the theorem follows. □

We saw in (3.7) that the upward passage time T_n^+ is a finite mixture of exponential variates. The fact that the exit rate of each state is constant, i.e., $\nu_n = V$, $n \in N$, enables one to show that T_n^+ can also be expressed as an infinite mixture of Gamma variates of odd order.

Theorem 4.6

T_n^+ and T_n^- for the Ehrenfest process are infinite mixtures of Gamma variates of odd order, $\Gamma(V, 2j+1)$, $j = 0,1,2,\ldots$.

Proof

The recurrence formula for $\sigma_n^+(s)$ in (3.2) can be rewritten as

$$\sigma_V^+(s) = \frac{r_n^+ \ \varepsilon(s)}{1 - r_n^- \ \varepsilon(s)\sigma_{n-1}^+(s)} \quad ; \quad \varepsilon(s) = \sigma_0^+(s) = \frac{V}{s+V} \tag{4.5}$$

where $r_n^+ = 1 - \frac{n}{2V}$ is the probability of going up given exit from n and $r_n^- = \frac{n}{2V}$ is that of going down. For Re(s) > 0, Eq. (4.5) has a series expansion

$$\sigma_n^+(s) = r_n^+\varepsilon(s) \sum_{j=0}^{\infty} \{r_n^-\varepsilon(s)\sigma_{n-1}^+(s)\}^j$$

and the result follows by induction for $\sigma_n^+(s)$. For $\sigma_n^-(s)$ it suffices to note that $\sigma_n^+(s) = \sigma_{2V-n}^-(s)$. □

§5. Convergence of the Ehrenfest process to the O-U process

Let $X_{OU}(t)$ be the O-U process with initial condition $X_{OU}(0) = x_0$. Then the state probability density of $X_{OU}(t)$ at $t > 0$ is given [ ] by

$$(5.1) \qquad g(x_0,x,t) = \frac{1}{\sqrt{2\pi(1 - e^{-2t})}} \exp\{- \frac{(x - x_0 e^{-t})^2}{2(1 - e^{-2t})}\} \quad , -\infty < x < \infty ,$$

i.e., $X_{OU}(t)$ is normally distributed with mean $x_0 e^{-t}$ and variance $1 - e^{-2t}$. The Laplace transform of $g(x_0,x,t)$ with respect to x is then given by

$$(5.2) \qquad \gamma(x_0,s,t) = \exp\{- x_0 e^{-t} s + \tfrac{1}{2}(1 - e^{-2t})s^2\} \quad .$$

We next show that the Ehrenfest process, suitably scaled and shifted, converges in distribution to $X_{OU}(t)$. Let $N_{2V}(t)$ be the Ehrenfest process as in §2. Let $X_V(t)$ be a process defined by

$$(5.3) \qquad X_V(t) = \sqrt{\frac{2}{V}}\, N_{2V}(t) - \sqrt{2V} \quad .$$

We note that $X_V(t)$ has discrete support on $\{r_0(t),\ldots,r_{2V}(t)\}$ where

$$(5.4) \qquad r_n(t) = \sqrt{\frac{2}{V}}\, n - \sqrt{2V} \ , \quad n - 0,1,2,\ldots \quad .$$

Clearly $r_{n+1}(t) - r_n(t) = \sqrt{\frac{2}{V}} \to 0$ as $V \to +\infty$. We note that the relaxation time of $X_V(t)$ is equal to 1 from (2.10) as is $X_{OU}(t)$. In the

following theorem we prove that when $N_{2V}(0)$ is chosen appropriately $X_V(t)$ converges in distribution to $X_{OU}(t)$ for every fixed t, $t \geq 0$, as $V \to \infty$. For notational convenience define

$$(5.5) \qquad \eta_V(x) = [\sqrt{\tfrac{V}{2}}\, x] \quad ,$$

where [x] is the minimum integer which is greater than or equal to x.

Theorem 5.1

Let $X_{OU}(t)$ be the O-U process with $X_{OU}(0) = x_0$, $-\infty < x_0 < \infty$. Let $X_V(t)$ be as in (5.3) with $X_V(0) = \sqrt{\frac{2}{V}}\,\eta_V(x_0)$ where V is chosen large enough so that $-\sqrt{2V} \leq X_V(0) \leq \sqrt{2V}$. Then $X_V(t) \xrightarrow{d} X_{OU}(t)$, for all t, $t \geq 0$, as $V \to +\infty$.

Proof

Let $\phi_V(x_0,w,t) = E[e^{-wX_V(t)} | X_V(0) = \sqrt{\frac{2}{V}}\,\eta_V(x_0)]$. One sees from (2.6) and (5.3) that

$$(5.6) \qquad \phi_V(x_0,w,t) = e^{w\sqrt{2V}}\, \beta_{N_{2V}(0)}(t,\, e^{-w\sqrt{\frac{2}{V}}})$$

where $N_{2V}(0) = V + \eta_V(x_0)$. We wish to show that $\phi_V(x_0,w,t) \to \gamma(x_0,w,t)$ as $V \to \infty$. Eq. (5.6) can be rewritten, by (2.7), as

$$\phi_V(x_0,w,t) = e^{w\sqrt{2V}}[(f(t) + g(t)e^{-w\sqrt{\frac{2}{V}}})(g(t) + f(t)e^{-w\sqrt{\frac{2}{V}}})]^V$$

$$\times \left[\frac{g(t) + f(t)e^{-w\sqrt{\frac{2}{V}}}}{f(t) + g(t)e^{-w\sqrt{\frac{2}{V}}}}\right]^{\eta_V(x_0)} .$$

Since f(t) + g(t) = 1, the first factor in the above equation becomes

$$e^{w\sqrt{2V}}\beta_V(t,\ e^{-w\sqrt{\frac{2}{V}}}) = [1 + 2f(t)g(t)\{\cosh(w\sqrt{\tfrac{2}{V}}) - 1\}]^V \quad .$$

For sufficiently small $|Re(w)|$, $\{f(t)g(t)|\cosh(w\sqrt{\frac{2}{V}}) - 1|\} < \frac{1}{2}$ so that

$$\log[e^{w\sqrt{2V}}\beta_V(t,\ e^{-w\sqrt{\frac{2}{V}}})] = V\log[1 + 2f(t)g(t)\{\cosh(w\sqrt{\tfrac{2}{V}}) - 1\}]$$

$$= V\sum_{k=1}^{\infty}\frac{1}{k}(2f(t)g(t))^k\{\cosh(w\sqrt{\tfrac{2}{V}}) - 1\}^k$$

$$= \frac{1}{2}(1 - e^{-2t})w^2 + O(V^{-1}) \quad .$$

The second factor can also be rewritten as

$$\left[1 - \frac{f(t) - g(t)\ (1 - e^{-w\sqrt{\frac{2}{V}}})}{f(t) + g(t)e^{-w\sqrt{\frac{2}{V}}}}\right]^{\eta_V(x_0)}$$

$$= \left[1 - \frac{e^{-t}w\sqrt{\frac{2}{V}} + O(V^{-1})}{1 + O(V^{-\frac{1}{2}})}\right]^{\eta_V(x_0)}$$

$$= \left[1 - \frac{x_0e^{-t}w + x_0O(V^{-\frac{1}{2}})}{x_0\sqrt{\frac{V}{2}}\{1 + O(V^{-\frac{1}{2}})\}}\right]^{x_0\sqrt{\frac{V}{2}}\cdot\frac{\eta_V(x_0)}{x_0\sqrt{\frac{V}{2}}}} \quad .$$

Therefore the second factor converges to $\exp\{-x_0e^{-t}w\}$ as $V \to \infty$. As a result, one has

$$\phi_V(x_0,w,t) \to \exp\{-x_0e^{-t}w + \frac{1}{2}(1 - e^{-2t})w^2\}$$

as $V \to \infty$, proving the theorem. □

The following corollary is immediate from Theorem 5.1.

<u>Corollary 5.2</u>

For given x_0, x real, let $m = V + \eta_V(x_0)$ and $n = V + \eta_V(x)$. Then

$$\sqrt{\frac{V}{2}}\, p_{mn}(t) \to g(x_0,x,t) \quad \text{as} \quad V \to \infty$$

for every t, $t \geq 0$.

Corollary 5.2 may be seen alternatively in the following manner. We first note that $g(x_0,x,t)$ in (5.1) has a series representation [ ]

$$(5.7) \qquad g(x_0,x,t) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \sum_{j=0}^{\infty} He_j(x_0)He_j(x)\, \frac{1}{j!}\, e^{-jt}$$

where $\{H_{2j}(x)\}_0^\infty$ is the set of the standardized Hermite polynomials. From (2.10) and Lemma 4.3, one obtains that

$$(5.8) \qquad \sqrt{\frac{V}{2}}\, p_{mn}(t) = \sqrt{\frac{V}{2}}\, \frac{\binom{2V}{n}}{2^{2V}} \sum_{j=0}^{2V} \binom{2V}{j} y_j(m)y_j(n)e^{-jt} \quad .$$

It is known (Szegö [11], p. 36) that

$$\lim_{V\to\infty} \sqrt{\binom{2V}{j}}\, y_j(n) = \frac{1}{\sqrt{j!}}\, He_j(x_0) \quad ,$$

and similarly

$$\lim_{V\to\infty} \sqrt{\binom{2V}{j}}\, y_j(n) = \frac{1}{\sqrt{j!}} He_j(x) \quad .$$

The first factor $\sqrt{\frac{V}{2}}\binom{2V}{n}/2^{2V}$ in (5.8) converges to $e^{-x^2/2}/\sqrt{2\pi}$ as $V \to \infty$ from Stirling's formula and $\sqrt{\frac{V}{2}}\, p_{mn}(t) \to g(x_0,x,t)$ as $V \to \infty$.

It is natural to expect that the first passage time distribution of $X_V(t)$ also converges in distribution to that of $X_{OU}(t)$. We next prove this formally.

Theorem 5.3

Let m and n be as in Lemma 5.2. Let $T_{r(m)r(n)}$ be the first passage time of $X_V(t)$ from r(m) to r(n) and let $T_{x_p x}$ be that of $X_{OU}(t)$ from x_0 to x. Then

$$T_{r(m)r(n)} \xrightarrow{d} T_{x_0 x} \quad \text{as} \quad V \to \infty \quad .$$

Proof

The Laplace transform of the p.d.f. of $T_{x_0 x}$ is given [ ] by

$$(5.9) \qquad E[e^{-sT_{x_0 x}}] = \frac{\gamma(x_0,x,s)}{\gamma(x,x,s)} \quad .$$

It is clear that $T_{r(m)r(n)} \stackrel{d}{=} T_{mn}$ where T_{mn} is the first passage time of $N_{2V}(t)$ from m to n. Let $s_{mn}(t)$ be the p.d.f. of T_{mn} with Laplace transform $\sigma_{mn}(s)$. From consistency relations, one has

$$(5.10) \qquad s_{mn}(t)*p_{nn}(t) = p_{mn}(t) \quad .$$

Let $\pi_{mn}(s)$ be the Laplace transform of $p_{mn}(t)$. One then has, from (5.10), that

$$\sigma_{mn}(s) = \frac{\pi_{mn}(s)}{\pi_{nn}(s)} = \frac{\sqrt{\frac{V}{2}}\,\pi_{mn}(s)}{\sqrt{\frac{V}{2}}\,\pi_{nn}(s)} \quad . \tag{5.11}$$

Hence from Corollary 5.2,

$$\sigma_{mn}(s) \rightarrow \frac{\gamma(x_0,x,s)}{\gamma(x,x,s)} \quad ,$$

proving the theorem. $\square$

§6. Numerical Results

In this section we compare some results for the Ehrenfest process with those obtained for the O-U process. The results for the O-U process are described in [8]. Those for the Ehrenfest process were derived in a two-step procedure. First, the analysis of the earlier sections was used to obtain numerical answers at the mass points of the Ehrenfest distribution. Then, Lagrange interpolation was used to extrapolate to the actual points desired.

Table 6.1 displays the ergodic cumulative distribution function $F_V(x) = P[X_V(\infty) \le x]$ for $V = 10, 25, 40$, and 80. The last column gives that of $X_{OU}(t)$, i.e., $F(x) = P[X_{OU}(\infty) \le x] = \int_0^x \{e^{-\frac{1}{2}y^2} \cdot \frac{1}{\sqrt{2\pi}}\}dy$. The interpolation procedure used here is as follows. For a given real x, we find an interval $[r(m), r(m+1))$ so that $r(m) \le x < r(m+1)$. Then Lagrange interpolation at the points $\{r(m-1), r(m), r(m+1), r(m+2)\}$ is used. As expected from the central limit theorem, the error here is quite small, less than 10^{-3} for $V = 80$.

In tables 6.2(a) and (b), the mean and standard deviation of the upward first passage time T_{OX} and the downward first passage time T_{XO} are shown. For each x, the corresponding row shows the mean of T_{OX} (or T_{XO}) for $X_V(t)$ with $V = 10, 25, 40, 80$ and for $X_{OU}(t)$. The second row shows the associated standard deviation. For $X_V(t)$, the recurrence formulas obtained from (3.2) (cf. Chapter 5 of [5]) are used and then an interpolation procedure as above. For the mean first passage time $E[T_{OX}]$, $V = 40$ gives 3 digit accuracy for $.1 \le x \le 2.2$, while $V = 80$ provides the same accuracy for $.1 \le x \le 2.6$. The relative error increases

as x grows larger than 3, probably due to numerical imprecision. For the mean downwards first passage time $E[T_{XO}]$, both V = 40 and V = 80 provide 3 digit accuracy for $.1 \le x \le 7.1$.

The remaining tables 6.3(a)-(e) compare the cumulative distribution functions of the first passage times for V = 10, 25, 40 and for $X_{OU}(t)$. Except for small values of t and small cumulative probabilities, agreement is quite good, even at V = 25. For example, the relative error for $S_{3,1}(\tau)$, $\tau \ge .75$ is less than 0.1% (for V = 40). For V = 25 the same error is less than 0.15%. For $S_{0,2}(\tau)$ the numbers are 1.7% for V = 40 and 2.7% for V = 25.

The greatest difficulty with the numerical procedures, described in the previous sections, occurs when attempting to find the zeros of the orthogonal polynomials. Of course, the major problem is that, even with the simplification allowed by Corollary 4.2, the polynomial $g_{80}(x)$ is of degree 40. The recursive definition of (3.4) may be used, at somewhat greater expense, as a more stable method of evaluation. Instability using a straightforward polynomial evaluation is detected by a failure to find all zeroes, thus indicating that the zero boundaries, i.e., the previous zeroes, are incorrect.

Once the zeroes are all found (a procedure which need be done only once), evaluation of passage time and exit time distributions is straightforward and relatively inexpensive.

Note: The missing data in table 6.2 is due to two factors. First, the state spaces do not cover the range completely for V = 10, 25, and 40. Second, some variance calculations exceeded the 10^{75} maximum real number on the computer.

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	$\chi_V(\infty)$				
x	10	25	40	80	$\chi_{OU}(\infty)$
0.00000	0.50000	0.50000	0.50000	0.50000	0.50000
0.20000	0.57772	0.57842	0.57870	0.57899	0.57847
0.40000	0.65149	0.65389	0.65440	0.65491	0.65415
0.60000	0.72102	0.72363	0.72441	0.72506	0.72445
0.80000	0.78239	0.78575	0.78666	0.78738	0.78719
1.00000	0.83564	0.83901	0.83987	0.84060	0.84088
1.20000	0.87971	0.88281	0.88360	0.88426	0.88489
1.40000	0.91476	0.91748	0.91815	0.91869	0.91944
1.60000	0.94179	0.94387	0.94437	0.94479	0.94544
1.80000	0.96174	0.96314	0.96351	0.96379	0.96424
2.00000	0.97565	0.97669	0.97691	0.97708	0.97733
2.20000	0.98534	0.98579	0.98591	0.98601	0.98610
2.50000	0.99352	0.99373	0.99375	0.99377	0.99377
2.80000	0.99745	0.99747	0.99746	0.99745	0.99743
3.00000	0.99873	0.99869	0.99867	0.99866	0.99865
3.20000	0.99941	0.99935	0.99934	0.99933	0.99932
3.50000	0.99983	0.99979	0.99978	0.99978	0.99977
3.80000	0.99995	0.99994	0.99994	0.99993	0.99993
4.00000	0.99999	0.99998	0.99997	0.99997	0.99997
4.30000	1.00000	0.99999	0.99999	0.99999	1.00000
4.60000	1.00000	1.00000	1.00000	1.00000	1.00000
5.00000	1.00000	1.00000	1.00000	1.00000	1.00000
6.00000	1.00000	1.00000	1.00000	1.00000	1.00000

Table 6.1. The ergodic cumulative distribution

x	$X_V(t)$ 10	25	40	80	$X_{OU}(t)$
1.0000 -1	3.9190 -1	6.4884 -2	9.9878 -2	1.2217 -1	1.3055 -1
	3.2359 0	0.0000 0	6.9620 -2	1.6019 -1	1.9065 -1
2.0000 -1	4.2376 -1	2.4612 -1	2.6674 -1	2.7128 -1	2.7248 -1
	2.2535 0	3.1277 -1	4.0158 -1	4.1673 -1	4.2056 -1
3.0000 -1	4.9407 -1	4.2235 -1	4.2326 -1	4.2606 -1	4.2739 -1
	1.5926 0	6.8453 -1	6.8313 -1	6.9730 -1	6.9966 -1
4.0000 -1	6.0283 -1	5.8724 -1	5.9280 -1	5.9483 -1	5.9720 -1
	1.2532 0	9.9327 -1	1.0282 0	1.0331 0	1.0408 0
5.0000 -1	7.8485 -1	7.7414 -1	7.7734 -1	7.8128 -1	7.8416 -1
	1.9231 0	1.4244 0	1.4313 0	1.4515 0	1.4608 0
6.0000 -1	1.0020 0	9.7759 -1	9.8296 -1	9.8754 -1	9.9099 -1
	2.8483 0	1.9175 0	1.9494 0	1.9715 0	1.9819 0
7.0000 -1	1.2225 0	1.2012 0	1.2112 0	1.2160 0	1.2209 0
	3.4017 0	2.5125 0	2.5879 0	2.6109 0	2.6336 0
8.0000 -1	1.4582 0	1.4596 0	1.4648 0	1.4729 0	1.4780 0
	3.8188 0	3.3718 0	3.3802 0	3.4367 0	3.4552 0
9.0000 -1	1.7236 0	1.7392 0	1.7548 0	1.7599 0	1.7668 0
	4.4481 0	4.2845 0	4.4523 0	4.4654 0	4.5004 0
1.0000 0	2.0608 0	2.0581 0	2.0730 0	2.0844 0	2.0934 0
	6.9835 0	5.5310 0	5.6787 0	5.7801 0	5.8420 0
1.2000 0	2.8383 0	2.8374 0	2.8600 0	2.8777 0	2.8902 0
	1.1026 1	9.1677 0	9.5154 0	9.7474 0	9.8567 0
1.3000 0	3.3003 0	3.3202 0	3.3528 0	3.3656 0	3.3804 0
	1.2905 1	1.2050 1	1.2612 1	1.2708 1	1.2867 1
1.6000 0	5.3949 0	5.3005 0	5.3449 0	5.3772 0	5.3988 0
	4.7614 1	2.7571 1	2.8596 1	2.9436 1	2.9800 1
2.0000 0	1.1064 1	1.0283 1	1.0384 1	1.0390 1	1.0428 1
	4.3982 2	9.5485 1	1.0398 2	1.0348 2	1.0527 2
2.2000 0	1.5098 1	1.4755 1	1.4873 1	1.4923 1	1.4935 1
	3.0310 2	1.9624 2	2.0883 2	2.1352 2	2.1448 2
2.6000 0	3.8511 1	3.2698 1	3.3363 1	3.3462 1	3.3472 1
	7.3992 3	6.0950 2	9.8385 2	1.0477 3	1.0848 3
3.0000 0	1.6165 2	8.6566 1	8.7087 1	8.8338 1	8.6931 1
	6.9182 5	3.5775 3	5.5132 3	7.6500 3	7.4243 3
3.5000 0	1.5634 3	3.5555 2	3.8293 2	3.7635 2	3.6436 2
			1.0113 5	1.2821 5	1.3202 5
4.0000 0			2.3792 3	2.1492 3	2.0184 3
			4.8987 6	3.5357 6	4.0691 6
5.0000 0			1.8053 5	1.7567 5	1.4074 5
				1.8622 10	1.9808 10
6.0000 0				5.1941 7	2.8268 7
				2.4841 15	7.9905 14
7.0000 0				4.2019 10	1.5980 10
					2.5536 20
8.0000 0					2.5148 13
					6.3240 26
9.0000 0					1.0947 17
					1.1985 34
1.0000 1					1.3130 27
					1.7240 42

Table 6.2(a). The mean and standard deviation of the upward first passage time T_{0x}

x	$X_V(t)$ 10	25	40	80	$X_{OU}(\infty)$
1.0000 -1	1.2307 -1	1.2024 -1	1.2012 -1	1.2025 -1	1.2053 -1
	4.3176 -1	4.0435 -1	4.0050 -1	3.9896 -1	3.9887 -1
2.0000 -1	2.3012 -1	2.3055 -1	2.3108 -1	2.3163 -1	2.3221 -1
	5.5216 -1	5.4151 -1	5.4076 -1	5.4080 -1	5.4099 -1
3.0000 -1	3.3060 -1	3.3343 -1	3.3440 -1	3.3521 -1	3.3603 -1
	6.3979 -1	6.3626 -1	6.3644 -1	6.3661 -1	6.3675 -1
4.0000 -1	4.2497 -1	4.2957 -1	4.3080 -1	4.3182 -1	4.3284 -1
	7.0793 -1	7.0760 -1	7.0778 -1	7.0788 -1	7.0796 -1
5.0000 -1	5.1379 -1	5.1953 -1	5.2097 -1	5.2218 -1	5.2339 -1
	7.6292 -1	7.6339 -1	7.6347 -1	7.6349 -1	7.6351 -1
6.0000 -1	5.9738 -1	6.0393 -1	6.0556 -1	6.0694 -1	6.0831 -1
	8.0794 -1	8.0825 -1	8.0821 -1	8.0817 -1	8.0812 -1
7.0000 -1	6.7609 -1	6.8331 -1	6.8512 -1	6.8664 -1	6.8817 -1
	8.4513 -1	8.4504 -1	8.4491 -1	8.4480 -1	8.4468 -1
8.0000 -1	7.5036 -1	7.5816 -1	7.6014 -1	7.6179 -1	7.6344 -1
	8.7622 -1	8.7567 -1	8.7547 -1	8.7530 -1	8.7511 -1
9.0000 -1	8.2058 -1	8.2891 -1	8.3102 -1	8.3279 -1	8.3457 -1
	9.0247 -1	9.0151 -1	9.0123 -1	9.0099 -1	9.0074 -1
1.0000 0	8.8710 -1	8.9591 -1	8.9815 -1	9.0003 -1	9.0191 -1
	9.2479 -1	9.2351 -1	9.2316 -1	9.2286 -1	9.2254 -1
1.2000 0	1.0103 0	1.0200 0	1.0224 0	1.0245 0	1.0266 0
	9.6055 -1	9.5875 -1	9.5826 -1	9.5784 -1	9.5741 -1
1.3000 0	1.0675 0	1.0776 0	1.0801 0	1.0823 0	1.0844 0
	9.7500 -1	9.7297 -1	9.7242 -1	9.7195 -1	9.7146 -1
1.6000 0	1.2245 0	1.2354 0	1.2382 0	1.2405 0	1.2428 0
	1.0088 0	1.0061 0	1.0054 0	1.0047 0	1.0041 0
2.0000 0	1.4054 0	1.4172 0	1.4202 0	1.4227 0	1.4252 0
	1.0388 0	1.0354 0	1.0345 0	1.0337 0	1.0329 0
2.2000 0	1.4861 0	1.4982 0	1.5013 0	1.5039 0	1.5065 0
	1.0496 0	1.0459 0	1.0450 0	1.0441 0	1.0433 0
2.6000 0	1.6319 0	1.6445 0	1.6477 0	1.6504 0	1.6531 0
	1.0660 0	1.0618 0	1.0607 0	1.0598 0	1.0589 0
3.0000 0	1.7605 0	1.7735 0	1.7768 0	1.7796 0	1.7824 0
	1.0775 0	1.0730 0	1.0718 0	1.0708 0	1.0697 0
3.5000 0	1.9023 0	1.9157 0	1.9191 0	1.9219 0	1.9247 0
	1.0875 0	1.0827 0	1.0814 0	1.0803 0	1.0792 0
4.0000 0	2.0274 0	2.0410 0	2.0445 0	2.0474 0	2.0503 0
	1.0946 0	1.0895 0	1.0881 0	1.0870 0	1.0858 0
5.0000 0		2.2541 0	2.2576 0	2.2606 0	2.2635 0
		1.0980 0	1.0966 0	1.0954 0	1.0941 0
6.0000 0		2.4307 0	2.4343 0	2.4373 0	2.4403 0
		1.1029 0	1.1015 0	1.1002 0	1.0989 0
7.0000 0		2.5813 0	2.5849 0	2.5880 0	2.5910 0
		1.1060 0	1.1045 0	1.1032 0	1.1019 0
8.0000 0			2.7162 0	2.7192 0	2.7223 0
			1.1066 0	1.1052 0	1.1039 0
9.0000 0				2.8354 0	2.8385 0
				1.1067 0	1.1053 0
1.0000 1				2.9396 0	2.9427 0
				1.1077 0	1.1063 0

Table 6.2(b). The mean and standard deviation of the downward first passage time T_{X0}

x	$X_V(t)$ 10	25	40	$X_{OU}(t)$
0.10	0.0000	0.0000	0.0000	0.0000
0.20	0.0000	0.0000	0.0000	0.0000
0.30	0.0000	0.0000	0.0000	0.0000
0.45	0.0002	0.0002	0.0002	0.0002
0.60	0.0007	0.0006	0.0006	0.0006
0.75	0.0014	0.0014	0.0014	0.0014
0.90	0.0023	0.0024	0.0025	0.0025
1.00	0.0030	0.0032	0.0033	0.0033
1.20	0.0046	0.0050	0.0051	0.0052
1.40	0.0064	0.0070	0.0071	0.0073
1.60	0.0083	0.0091	0.0093	0.0095
2.00	0.0121	0.0133	0.0136	0.0141
2.20	0.0141	0.0155	0.0158	0.0163
2.60	0.0180	0.0198	0.0203	0.0206
3.00	0.0219	0.0241	0.0247	0.0255
3.40	0.0258	0.0284	0.0291	0.0300
3.80	0.0297	0.0327	0.0335	0.0345
5.00	0.0413	0.0454	0.0464	0.0479

Table 6.3(a). $S_{03}(x) = P[T_{03} \le x]$

x	$X_V(t)$ 10	25	40	$X_{OU}(t)$
0.1000	0.0001	0.0000	0.0000	0.0000
0.2000	0.0004	0.0003	0.0003	0.0002
C.3000	0.0014	0.0013	0.0013	0.0012
0.4500	0.0037	0.0040	0.0040	0.0041
0.6000	0.0066	0.0073	0.0075	0.0077
0.7500	0.0097	0.0108	0.0111	0.0115
0.9000	0.0126	0.0141	0.0145	0.0151
1.0000	0.0145	0.0162	0.0167	0.0174
1.2000	0.0180	0.0202	0.0207	0.0216
1.4000	0.0212	0.0238	0.0244	0.0254
1.6000	0.0242	0.0271	0.0278	0.0289
2.0000	0.0296	0.0330	0.0339	0.0351
2.2000	0.0320	0.0357	0.0366	0.0380
2.6000	0.0367	0.0408	0.0418	0.0433
3.0000	0.0410	0.0455	0.0467	0.0483
3.4000	0.0451	0.0500	0.0513	0.0531
3.8000	0.0491	0.0544	0.0558	0.0577
5.0000	0.0607	0.0671	0.0687	0.0710

Table 6.3(b). $S_{13}(x) = P[T_{13} \le x]$

x	$X_V(t)$ 10	25	40	$X_{OU}(t)$
0.10	0.0080	0.0081	0.0076	0.0067
0.20	0.0242	0.0272	0.0279	0.0288
0.30	0.0389	0.0445	0.0459	0.0479
0.45	0.0557	0.0636	0.0656	0.0684
0.60	0.0675	0.0767	0.0790	0.0822
0.75	0.0763	0.0863	0.0888	0.0922
0.90	0.0830	0.0935	0.0962	0.0998
1.00	0.0866	0.0975	0.1002	0.1040
1.20	0.0927	0.1040	0.1069	0.1108
1.40	0.0975	0.1092	0.1122	0.1162
1.60	0.1015	0.1135	0.1166	0.1207
2.00	0.1080	0.1206	0.1237	0.1280
2.20	0.1108	0.1235	0.1268	0.1311
2.60	0.1157	0.1288	0.1322	0.1367
3.00	0.1200	0.1336	0.1370	0.1416
3.40	0.1241	0.1379	0.1414	0.1462
3.80	0.1279	0.1421	0.1456	0.1505
5.00	0.1387	0.1538	0.1576	0.1628

Table 6.3(c). $S_{23}(x) = P[T_{23} \leq x]$

x	$X_V(t)$ 10	25	40	$X_{OU}(t)$
0.10	0.0001	0.0000	0.0000	0.0000
0.20	0.0019	0.0011	0.0009	0.0006
0.30	0.0062	0.0048	0.0044	0.0038
0.45	0.0170	0.0152	0.0147	0.0140
0.60	0.0309	0.0290	0.0286	0.0278
0.75	0.0461	0.0443	0.0438	0.0431
0.90	0.0617	0.0599	0.0594	0.0588
1.00	0.0720	0.0702	0.0698	0.0691
1.20	0.0923	0.0905	0.0901	0.0894
1.40	0.1119	0.1100	0.1096	0.1089
1.60	0.1308	0.1288	0.1284	0.1276
2.00	0.1665	0.1644	0.1639	0.1631
2.20	0.1835	0.1813	0.1808	0.1800
2.60	0.2161	0.2137	0.2132	0.2123
3.00	0.2470	0.2445	0.2439	0.2430
3.40	0.2764	0.2738	0.2732	0.2722
3.80	0.3046	0.3018	0.3012	0.3002
5.00	0.3824	0.3793	0.3786	0.3775

Table 6.3(d). $S_{02}(x) = P[T_{02} \leq x]$

x	$X_V(t)$ 10	25	40	$X_{OU}(t)$
0.10	0.0203	0.0160	0.0146	0.0119
0.20	0.0591	0.0559	0.0551	0.0538
0.30	0.0955	0.0940	0.0938	0.0934
0.45	0.1399	0.1397	0.1397	0.1398
0.60	0.1744	0.1746	0.1747	0.1749
0.75	0.2021	0.2024	0.2025	0.2027
0.90	0.2253	0.2256	0.2256	0.2258
1.00	0.2389	0.2391	0.2392	0.2393
1.20	0.2628	0.2630	0.2630	0.2631
1.40	0.2837	0.2837	0.2837	0.2837
1.60	0.3023	0.3022	0.3022	0.3022
2.00	0.3351	0.3348	0.3347	0.3346
2.20	0.3499	0.3495	0.3494	0.3493
2.60	0.3775	0.3768	0.3767	0.3765
3.00	0.4029	0.4021	0.4019	0.4017
3.40	0.4268	0.4258	0.4256	0.4253
3.80	0.4494	0.4483	0.4480	0.4477
5.00	0.5113	0.5098	0.5095	0.5090

Table 6.3(e). $S_{12}(x) = P[T_{12} \leq x]$

x	$X_V(t)$ 10	25	40	$X_{OU}(t)$
0.10	0.0386	0.0285	0.0257	0.0204
0.20	0.1103	0.1005	0.0981	0.0941
0.30	0.1787	0.1710	0.1692	0.1663
0.45	0.2640	0.2579	0.2564	0.2539
0.60	0.3319	0.3263	0.3249	0.3226
0.75	0.3876	0.3821	0.3807	0.3785
0.90	0.4346	0.4292	0.4278	0.4256
1.00	0.4623	0.4569	0.4556	0.4534
1.20	0.5113	0.5060	0.5047	0.5025
1.40	0.5536	0.5483	0.5470	0.5449
1.60	0.5908	0.5857	0.5844	0.5823
2.00	0.6540	0.6491	0.6479	0.6459
2.20	0.6812	0.6765	0.6753	0.6733
2.60	0.7287	0.7243	0.7232	0.7213
3.00	0.7687	0.7646	0.7636	0.7619
3.40	0.8027	0.7989	0.7979	0.7963
3.80	0.8315	0.8281	0.8272	0.8257
5.00	0.8951	0.8925	0.8918	0.8907

Table 6.3(f). $S_{01}(x) = P[T_{01} \leq x]$

x	$X_V(t)$ 10	25	40	$X_{OU}(t)$
0.1000	0.1294	0.1028	0.0951	0.0808
0.2000	0.3501	0.3406	0.3389	0.3367
0.3000	0.5384	0.5413	0.5424	0.5447
0.4500	0.7308	0.7368	0.7383	0.7408
0.6000	0.8436	0.8477	0.8487	0.8503
0.7500	0.9088	0.9111	0.9116	0.9125
0.9000	0.9466	0.9478	0.9480	0.9484
1.0000	0.9626	0.9633	0.9634	0.9637
1.2000	0.9816	0.9818	0.9818	0.9819
1.4000	0.9909	0.9909	0.9909	0.9909
1.6000	0.9955	0.9955	0.9954	0.9940
2.0000	0.9989	0.9989	0.9989	0.9988
2.2000	0.9995	0.9994	0.9994	0.9999
2.6000	0.9999	0.9999	0.9999	0.9999
3.0000	1.0000	1.0000	1.0000	1.0000
3.4000	1.0000	1.0000	1.0000	1.0000
3.8000	1.0000	1.0000	1.0000	1.0000
5.0000	1.0000	1.0000	1.0000	1.0000

Table 6.3(g). $S_{-3-2}(x) = P[T_{-3,-2} \leq x]$

x	$X_V(t)$ 10	25	40	$X_{OU}(t)$
0.10	0.0025	0.0007	0.0004	0.0001
0.20	0.0268	0.0173	0.0148	0.0103
0.30	0.0857	0.0718	0.0680	0.0612
0.45	0.2196	0.2086	0.2058	0.2011
0.60	0.3689	0.3635	0.3622	0.3602
0.75	0.5060	0.5045	0.5042	0.5038
0.90	0.6209	0.6214	0.6216	0.6219
1.00	0.6845	0.6856	0.6859	0.6864
1.20	0.7839	0.7852	0.7856	0.7862
1.40	0.8532	0.8545	0.8548	0.8553
1.60	0.9009	0.9018	0.9020	0.9024
2.00	0.9551	0.9556	0.9557	0.9559
2.20	0.9699	0.9702	0.9703	0.9704
2.60	0.9864	0.9866	0.9866	0.9867
3.00	0.9939	0.9940	0.9940	0.9940
3.40	0.9973	0.9973	0.9973	0.9973
3.80	0.9988	0.9988	0.9988	0.9988
5.00	0.9999	0.9999	0.9999	0.9999

Table 6.3(h). $S_{-3-1}(x) = P[T_{-3,-1} \leq x]$

x	$X_V(t)$ 10	25	40	$X_{OU}(t)$
0.10	0.0012	0.0003	0.0002	0.0000
0.20	0.0133	0.0079	0.0066	0.0043
0.30	0.0433	0.0341	0.0316	0.0274
0.45	0.1160	0.1053	1025	0.0978
0.60	0.2051	0.1956	0.1932	0.1892
0.75	0.2968	0.2890	0.2871	0.2838
0.90	0.3840	0.3777	0.3761	0.3735
1.00	0.4379	0.4325	0.4311	0.4288
1.20	0.5346	0.5304	0.5293	0.5276
1.40	0.6163	0.6130	0.6122	0.6108
1.60	0.6844	0.6819	0.6812	0.6801
2.00	0.7875	0.7858	0.7854	0.7847
2.20	0.8258	0.8245	0.8241	0.8236
2.60	0.8831	0.8822	0.8820	0.8816
3.00	0.9216	0.9210	0.9208	0.9206
3.40	0.9474	0.9470	0.9469	0.9468
3.80	0.9647	0.9645	0.9644	0.9643
5.00	0.9894	0.9893	0.9893	0.9892

Table 6.3(i). $S_{-2,0}(x) = P[T_{-2,0} \le x]$

x	$X_V(t)$ 10	25	40	$X_{OU}(t)$
0.10	0.0629	0.0465	0.0420	0.0336
0.20	0.1773	0.1632	0.1597	0.1539
0.30	0.2853	0.2759	0.2737	0.2701
0.45	0.4171	0.4115	0.4101	0.4078
0.60	0.5183	0.5142	0.5132	0.5115
0.75	0.5973	0.5941	0.5933	0.5920
0.90	0.6607	0.6581	0.6574	0.6563
1.00	0.6963	0.6939	0.6934	0.6924
1.20	0.7552	0.7534	0.7529	0.7521
1.40	0.8016	0.8001	0.7998	0.7991
1.60	0.8387	0.8375	0.8372	0.8367
2.00	0.8927	0.8919	0.8917	0.8914
2.20	0.9123	0.9117	0.9115	0.9112
2.60	0.9414	0.9409	0.9408	0.9406
3.00	0.9607	0.9604	0.9604	0.9602
3.40	0.9737	0.9735	0.9734	0.9734
3.80	0.9824	0.9822	0.9822	0.9821
5.00	0.9947	0.9947	0.9946	0.9946

Table
Table 6.3(j). $S_{-1,0}(x) = P[T_{-1,0} \le x]$

	$X_V(t)$			
x	10	25	40	$X_{OU}(t)$
0.10	0.0000	0.0000	0.0000	0.0000
0.20	0.0005	0.0002	0.0001	0.0000
0.30	0.0042	0.0021	0.0017	0.0009
0.45	0.0233	0.0172	0.0157	0.0130
0.60	0.0640	0.0552	0.0529	0.0489
0.75	0.1243	0.1146	0.1121	0.1079
0.90	0.1972	0.1881	0.1858	0.1819
1.00	0.2494	0.2410	0.2389	0.2353
1.20	0.3547	0.3479	0.3462	0.3433
1.40	0.4541	0.4488	0.4475	0.4452
1.60	0.5432	0.5391	0.5380	0.5363
2.00	0.6863	0.6837	0.6831	0.6820
2.20	0.7415	0.7394	0.7389	0.7380
2.60	0.8255	0.8241	0.8238	0.8232
3.00	0.8826	0.8817	0.8815	0.8611
3.40	0.9212	0.9206	0.9205	0.9202
3.80	0.9471	0.9467	0.9466	0.9465
5.00	0.9841	0.9840	0.9839	0.9839

Table 6.3(k). $S_{-3,0}(x) = P[T_{-3,0} \leq x]$

	$X_V(t)$			
x	10	25	40	$X_{OU}(t)$
0.10	0.0932	0.0711	0.0648	0.0531
0.20	0.2582	0.2440	0.2406	0.2353
0.30	0.4084	0.4032	0.4021	0.4006
0.45	0.5801	0.5802	0.5803	0.5806
0.60	0.6992	0.7006	0.7010	0.7016
0.75	0.7825	0.7840	0.7844	0.7850
0.90	0.8416	0.8429	0.8432	0.8438
1.00	0.8713	0.8725	0.8728	0.8733
1.20	0.9147	0.9156	0.9158	0.9161
1.40	0.9432	0.9438	0.9440	0.9442
1.60	0.9621	0.9625	0.9626	0.9628
2.00	0.9831	0.9832	0.9833	0.9834
2.20	0.9887	0.9888	0.9888	0.9889
2.60	0.9949	0.9950	0.9950	0.9950
3.00	0.9977	0.9977	0.9977	0.9978
3.40	0.9990	0.9990	0.9990	0.9990
3.80	0.9995	0.9995	0.9995	0.9995
5.00	1.0000	1.0000	1.0000	1.0000

Table 6.3(ℓ). $S_{-2,-1}(x) = P[T_{-2,-1} \leq x]$

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